Dynamical Behavior of Nonautonomous Stochastic Reaction-Diffusion Neural Network Models

Wei, Tengda; Lin, Ping; Zhu, Quanxin ; Wang, Linshan; Wang, Yangfan

Published in:
IEEE Transactions on Neural Networks and Learning Systems

DOI:
10.1109/TNNLS.2018.2869028

Publication date:
2019

Accepted Manuscript. Final version available via DOI: 10.1109/TNNLS.2018.2869028

Dynamical Behavior of Nonautonomous Stochastic Reaction-Diffusion Neural Network Models

Tengda Wei, Ping Lin, Quanxin Zhu, *Senior Member, IEEE,* Linshan Wang, *Member, IEEE* and Yangfan Wang

***Abstract*—This brief paper investigates nonautonomous stochastic reaction-diffusion neural network models with S-type distributed delays. First, the existence and uniqueness of mild solution are studied under the Lipschitz condition without the linear growth condition. Due to the existence of a nonautonomous reaction-diffusion term and the infinite dimensional Wiener process, the criteria for the well-posedness of the models are established based on the evolution system theory. Then, the S-type distributed delay, which is an infinite delay, is handled by the truncation method and sufficient conditions for the global exponential stability are obtained by constructing a simple Lyapunov-Krasovskii functional candidate. Finally, neural network examples and an illustrative example are given to show the applications of the obtained results.**

***Index Terms*—Stochastic neural network, Reaction-diffusion, S-type delay, Mild solution, Existence-uniqueness and stability.**

I. Introduction

In the past decades, neural networks have been intensely investigated due to their wide applications in our life, such as communication networks, social networks, power grids, cellular networks, and World Wide Web. The dynamical behavior of neural networks can well explain real phenomena and provide fundamental support for application. Therefore, many researchers have studied dynamical properties of neural networks. Such studies include the existence-uniqueness of solutions [1], periodic solutions [2], [3], stability of equilibrium point [4], [5], synchronization [6], [7], etc.

In real world, the diffusion effect widely exists in biological and artificial neural networks. For instance, the dynamical behavior of multilayer cellular neural networks heavily depends on both time and position of variables, and the interactions arising from the space-distributed structure of the whole networks can be seen as a diffusion phenomenon [8]. Hence, reaction-diffusion neural networks (RDNNs) have attracted the attention of quite a number of researchers (see e.g. [9], [10]). The reaction-diffusion term can be treated as a linear operator in a proper Hilbert space so that a RDNN is transformed to a time dependent functional differential equation with a spatial differential operator defined in the Hilbert space and criteria of well-posedness and stability are established in the Hilbert space instead of an Euclidean Space [1]. On the other hand, time delay often occurs in the electronic implementation of analog networks due to the finite speed of signal transmission and amplifier switching. Hence, discrete time delays or distributed time delays are included into RDNNs [4], [11], [12].

The stochastic perturbation is unavoidable in the propagation of electric potential in a neuron because the signals released by neurotransmitters may have random fluctuation in time. Therefore, it is natural to assume that the system is perturbed by a finite dimensional Brownian motion [13], [14]. However, the neurons, as long thin cylinders, actually act like electrical cables with a spatial dimension, so the random noise is preferable to be modelled as an infinite dimensional Wiener process and depends on both time and space [1]. Thus, there have been several papers about stochastic RDNNs (SRDNNs) driven by the infinite dimensional Wiener process [15], [16].

Most of these works have dealt with autonomous models, but nonautonomous phenomena often occur in real neural networks. The parameters of SRDNNs may change along with time, resulting in an essential change of the system. For instance, the autonomous reaction-diffusion term generates a continuous semigroup while the nonautonomous one does not. Although there have been works about SRDNNs and nonautonomous SNNs (see e.g. [4], [13], [14], [17]–[19]), few authors have considered dynamical behavior of nonautonomous SRDNNs with nonautonomous reaction-diffusion terms, especially the existence-uniqueness of the mild solution which is fundamental for further researches such as stability and synchronization. Inspired by the aforementioned discussion, we investigate a nonautonomous stochastic reaction-diffusion differential system with an S-type distributed delay and establish criteria for the existence-uniqueness and global stability of mild solution. The novelty of this brief lies in the following aspects: (1) the existence-uniqueness theorem is established based on the evolution system theory in a Hilbert space due to a nonautonomous reaction-diffusion term and the infinite dimensional Wiener process; (2) the mild solution can not blow up in finite time under the Lipschitz condition and a weaker linear growth condition; (3) the global exponential stability in the mean-square sense of the system with infinite delay is obtained by the truncation method and

Manuscript received Month Day, Year; revised Month Day, Year. This work was supported by the National Natural Science Foundation of China (11771014, 61773217, 61374080, 91430106), the Natural Science Foundation of Jiangsu Province (BK20161552), the Alexander von Humboldt Foundation of Germany (Fellowship CHN/1163390), Fundamental Research Funds for the Central Universities (No. 06500073), and China Scholarship Council (File No. 201706330011). (Corresponding authors: Ping Lin and Yangfan Wang.)

T. Wei is with the College of Oceanic and Atmospheric Sciences, Ocean University of China, Qingdao 266100, China (e-mail: tdwei123@163.com).

P. Lin and T. Wei are with the Department of Mathematics, University of Dundee, Dundee DD1 4HN, UK (e-mail: plin@maths.dundee.ac.uk).

Q. Zhu is with the School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing 210023, China, and also with the Department of Mathematics, University of Bielefeld, Bielefeld D-33615, Germany (e-mail: zqx22@126.com).

L. Wang is with the School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China (e-mail: wangls@ouc.edu.cn).

Y. Wang is with the College of Marine Life Science, Ocean University of China, Qingdao 266100, China (e-mail: yfwang@ouc.edu.cn).

the construction of a simple Lyapunov-Krasovskii functional candidate without distributed term. The obtained results are easily applied to many neural network models, such as Hopfield neural networks and Cohen-Grossberg neural networks.

Notations: $\mathfrak{L} = (L^2(\mathbb{O}))^n$ and $\mathfrak{H} = (H_0^1(\mathbb{O}))^n$ are Hilbert spaces with norms $\|\boldsymbol{u}\| = (\sum_{i=1}^n \int_{\mathbb{O}} |u_i|^2 d\boldsymbol{x})^{\frac{1}{2}}$ and $\|\boldsymbol{u}\|_{\mathfrak{H}} = (\sum_{i=1}^n \sum_{j=1}^l \int_{\mathbb{O}} (\frac{\partial u_i}{\partial x_j})^2 d\boldsymbol{x})^{\frac{1}{2}}$. $\mathfrak{C} = C((-\infty, t_0], \mathfrak{L})$ represents the space of continuous bounded functions from $(-\infty, t_0]$ to $\mathfrak{L}$ with norm $\|\boldsymbol{\varphi}\|_{\mathfrak{C}} = \sup_{-\infty \le \theta \le t_0} \|\boldsymbol{\varphi}(\theta)\|$. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with filtration $\{\mathcal{F}_t\}_{t \ge t_0}$. For any continuous $\mathcal{F}_t$-adapted $\mathfrak{L}$-valued stochastic process $\boldsymbol{u}(t, \boldsymbol{x})(\omega) : \Omega \to \mathfrak{L}$, we define a continuous $\mathcal{F}_t$-adapted $\mathfrak{C}$-valued stochastic process $\boldsymbol{u}_t(\omega) = \boldsymbol{u}(t+\theta, \boldsymbol{x})(\omega)$ and $\|\boldsymbol{u}_t\|_{\mathfrak{C}} = \sup_{-\infty \le \theta \le 0} \mathcal{E}\|\boldsymbol{u}(t+\theta)\|$ where $\mathcal{E}(\cdot)$ is the expectation operator. $\mathfrak{C}^b_{\mathcal{F}_0}$ denotes the family of $\mathcal{F}_0$-measurable bounded stochastic variables $\boldsymbol{\phi}$ with $\mathcal{E}\|\boldsymbol{\phi}\|_{\mathfrak{C}} < \infty$. $\|\boldsymbol{A}\|_F \triangleq (\mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^T))^{\frac{1}{2}}$ is the Frobenius norm of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and tr is the trace operator. $\mathscr{L}_2^0(\Re, \mathfrak{L})$ describes the space of Hilbert-Schmidt operators from $\Re \triangleq Q^{\frac{1}{2}}((L^2(\mathbb{O}))^m)$ into $\mathfrak{L}$ where Q is a positive definite, self-adjoint, Hilbert-Schmidt operator with a finite trace and $\|\Psi\|_* \triangleq \sqrt{\mathrm{tr}(\Psi Q \Psi^*)}$.

II. Problem formulation and preliminaries

Consider the following nonautonomous stochastic functional differential system with S-type distributed delays

$$\begin{cases} d\boldsymbol{u}(t,\boldsymbol{x}) = [\nabla \cdot (\boldsymbol{D}(t,\boldsymbol{x}) \circ \nabla \boldsymbol{u}(t,\boldsymbol{x})) + \boldsymbol{C}(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x}) \\ \qquad\qquad + \boldsymbol{f}(t, \mathcal{S}(\boldsymbol{u}))]dt + \boldsymbol{G}(t, \mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(t,\boldsymbol{x}), \\ \boldsymbol{u}(t,\boldsymbol{x}) \mid_{\boldsymbol{x} \in \partial\mathbb{O}} = 0, \ t \ge t_0, \\ \boldsymbol{u}(\theta, \boldsymbol{x}) = \boldsymbol{\varphi}(\theta, \boldsymbol{x}) \in \mathfrak{C}^b_{\mathcal{F}_0}, \ -\infty \le \theta \le t_0, \ \boldsymbol{x} \in \mathbb{O}, \end{cases} \tag{1}$$

where $\boldsymbol{u}(t,\boldsymbol{x}) = (u_1(t,\boldsymbol{x}), u_2(t,\boldsymbol{x}), \cdots, u_n(t,\boldsymbol{x}))^T$, $\boldsymbol{f} = (f_1, f_2, \cdots, f_n)^T$, and $\boldsymbol{G} = (G_{ij})_{n \times m} \in M_2^{n,m}(t_0, t)$ [20]. Besides, $\boldsymbol{C}(t,\boldsymbol{x}) = diag(c_1(t,\boldsymbol{x}), c_2(t,\boldsymbol{x}), \cdots, c_n(t,\boldsymbol{x}))$, $\boldsymbol{D}(t,\boldsymbol{x}) = (D_{ij}(t,\boldsymbol{x}))_{n \times l}$, $\nabla \cdot (\boldsymbol{D}(t,\boldsymbol{x}) \circ \nabla \boldsymbol{u}) = (\sum_{j=1}^l \frac{\partial (D_{1j}(t,\boldsymbol{x}) \frac{\partial u_1}{\partial x_j})}{\partial x_j}, \cdots, \sum_{j=1}^l \frac{\partial (D_{nj}(t,\boldsymbol{x}) \frac{\partial u_n}{\partial x_j})}{\partial x_j})^T$, and $\circ$ denotes Hadamard product [21]. $\mathcal{S}(\boldsymbol{u}) = \int_{-\infty}^0 d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta, \boldsymbol{x})$ is Lebesgue-Stieltjes integrable and $\boldsymbol{\eta}(\theta)$ is non-decreasing bounded variation function which satisfies $\int_{-\infty}^0 d\boldsymbol{\eta}(\theta) = (\hat{\eta}_{ij})_{n \times n}$, $\hat{\eta}_{ij} > 0$. $\mathbb{O}$ is an open bounded and connected subset of R^l with a sufficiently regular boundary $\partial\mathbb{O}$. $\boldsymbol{W} = \sum_{n=1}^\infty \sqrt{\alpha_n}\beta_n(t)e_n(x)$ is a spatiotemporal Wiener process with values in a separable Hilbert space where $\sum_{n=1}^\infty \alpha_n < +\infty$, $\{e_n\}_{n=1}^\infty$ is an orthogonal basis of $\mathfrak{L}$ and $\{\beta_n\}_{n=1}^\infty$ is a sequence of mutually independent standard Brownian motions [20].

Let us define the linear operator $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}(t)) \to \mathfrak{L}$, $\mathcal{A}(t)\boldsymbol{u}(t,\boldsymbol{x}) = \nabla \cdot (\boldsymbol{D}(t,\boldsymbol{x}) \circ \nabla \boldsymbol{u}(t,\boldsymbol{x})) + \boldsymbol{C}(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x})$ where $\mathcal{D}(\mathcal{A}(t)) = (H^2(\mathbb{O}))^n \cap (\mathring{H}_0^1(\mathbb{O}))^n \subset \mathfrak{L}$. $\mathring{H}_0^1(\mathbb{O})$ is the closure of $C_0^\infty(\mathbb{O})$ in $H_0^1(\mathbb{O})$ and $C_0^\infty(\mathbb{O})$ is the set of all infinitely differentiable functions with a compact support in $\mathbb{O}$. Then, we rewrite (1) in the following equivalence form

$$\begin{cases} d\boldsymbol{u}(t) = [\mathcal{A}(t)\boldsymbol{u}(t) + \boldsymbol{f}(t, \mathcal{S}(\boldsymbol{u}))]dt \\ \qquad\qquad + \boldsymbol{G}(t, \mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}, \ t > t_0 \\ \boldsymbol{u}(t) = \boldsymbol{\varphi}(t) \in \mathfrak{C}^b_{\mathcal{F}_0}, \ t \le t_0. \end{cases} \tag{2}$$

In this brief, we shall assume that

H1: $D_{ij}(t,\boldsymbol{x})$ and $c_i(t,\boldsymbol{x})$ are real-valued smooth functions with bounded derivatives, $D_{ij}(t,\boldsymbol{x}) \ge \alpha$, $c_i(t,\boldsymbol{x}) \le -\beta$ and $\beta > \alpha/2$, where $\alpha, \beta > 0$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, l$, $t \ge t_0$, and $\boldsymbol{x} \in \mathbb{O}$;

H2: $\boldsymbol{f}$ and $\boldsymbol{G}$ satisfy the Lipschitz condition: $\|\boldsymbol{f}(t,\boldsymbol{u}) - \boldsymbol{f}(t,\boldsymbol{v})\| \vee \|\boldsymbol{G}(t,\boldsymbol{u}) - \boldsymbol{G}(t,\boldsymbol{v})\| \vee \|\boldsymbol{G}(t,\boldsymbol{u}) - \boldsymbol{G}(t,\boldsymbol{v})\|_* \le L\|\boldsymbol{u} - \boldsymbol{v}\|$, where $t \ge t_0$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{L}$;

H3: For any $T \ge t_0$, there exists $m(T) > 0$ such that $\|\boldsymbol{f}(t,\boldsymbol{0})\| \vee \|\boldsymbol{G}(t,\boldsymbol{0})\| \vee \|\boldsymbol{G}(t,\boldsymbol{0})\|_* \le m(T)$, $\forall t \in [t_0, T]$.

Under H1, $\mathcal{A}(\cdot)$ is the infinitesimal generator of an evolution system $(U(t,s))_{t,s \ge t_0}$ (see [22]).

Definition 1 ([22]). A predictable $\mathfrak{L}$-valued process $\boldsymbol{u}(t)$ is said to be a mild solution of (1), if $\mathbb{P}(\int_{t_0}^t \|\boldsymbol{u}(s,\boldsymbol{x})\|^2 ds < +\infty) = 1$ and

$$\begin{aligned} \boldsymbol{u}(t,\boldsymbol{x}) = U(t,t_0)\boldsymbol{\varphi}(t_0) + \int_{t_0}^t U(t,s)\boldsymbol{f}(s, \mathcal{S}(\boldsymbol{u}))ds \\ + \int_{t_0}^t U(t,s)\boldsymbol{G}(s, \mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(s,\boldsymbol{x}), \end{aligned} \tag{3}$$

where $t \in [t_0, +\infty)$.

Definition 2. The equilibrium point $\boldsymbol{u}^*$ to equation (1) is globally exponentially stable in the mean-square sense if, for any solution $\boldsymbol{u}$ with the initial condition $\boldsymbol{\varphi} \in \mathfrak{C}^b_{\mathcal{F}_0}$, there exist positive constants M and ϵ such that

$$\mathcal{E}\|\boldsymbol{u} - \boldsymbol{u}^*\|^2 \le Me^{-\epsilon(t-t_0)}\mathcal{E}\|\boldsymbol{\varphi} - \boldsymbol{u}^*\|^2_{\mathfrak{C}}, \ t \ge t_0. \tag{4}$$

Lemma 1 (Poincaré inequallity [1]). Let $\mathbb{O}$ be an open bounded smooth domain in $\mathbb{R}^l$, then $\|\boldsymbol{u}\| \le \kappa^{-1}\|\boldsymbol{u}\|_{\mathfrak{H}}$, where $\boldsymbol{u} \in \mathfrak{H}$ and the constant $\kappa > 0$ depends on the size of domain $\mathbb{O}$.

Lemma 2 ([21]). Let $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_n(t))^T$, $\boldsymbol{\varsigma}(t) = (\varsigma_1(t), \varsigma_2(t), \cdots, \varsigma_n(t))^T$, $\bar{\boldsymbol{\eta}}(t) = \sup_{-\tau \le \theta \le 0} \boldsymbol{\eta}(t+\theta)$, and $\bar{\boldsymbol{\varsigma}}(t) = \sup_{-\tau \le \theta \le 0} \boldsymbol{\varsigma}(t+\theta)$. Then $\boldsymbol{\eta}(t) \le \boldsymbol{\varsigma}(t)$ for $t \ge t_0$, if the following conditions hold

$$\begin{aligned} &(L_1) \ \boldsymbol{\eta}(t) < \boldsymbol{\varsigma}(t), \ t_0 - \tau \le t \le t_0; \\ &(L_2) \ D^+\boldsymbol{\varsigma}(t) > \boldsymbol{\Phi}(t, \boldsymbol{\varsigma}(t), \bar{\boldsymbol{\varsigma}}(t)), \ t \ge t_0; \\ &(L_3) \ D^+\boldsymbol{\eta}(t) \le \boldsymbol{\Phi}(t, \boldsymbol{\eta}(t), \bar{\boldsymbol{\eta}}(t)), \ t \ge t_0, \end{aligned}$$

where $\boldsymbol{\Phi}(t, \boldsymbol{x}, \boldsymbol{y})$ is an M-function.

III. Existence and uniqueness of the mild solution

In this section, we shall establish the existence-uniqueness theorem of the mild solution to (1). First, some propositions about the mild solution are derived.

Proposition 1. Assume that H1-H3 hold. Then, there exists $T > t_0$ such that equation (1) has a unique mild solution $\boldsymbol{u}(t,\boldsymbol{x})$ for $t \in [t_0, T]$.

Proof. Since $U(t,s)$ is the evolution operator of $\mathcal{A}(t)$, then $(t,s) \to U(t,s)$ is strongly continuous for $t_0 \le s \le t \le +\infty$. Let $N(T) \triangleq \sup_{t_0 \le s \le t \le T} \|U(t,s)\|$. Choose T close to t_0 such that

$$2nL^2\|\hat{\boldsymbol{\eta}}\|_F^2 N^2(T)(T - t_0)(T - t_0 + 1) < 1. \tag{5}$$

We denote the Banach space of all the $\mathfrak{L}$-valued predictable process $\boldsymbol{u}(t,\boldsymbol{x})$ by $\mathfrak{L}_T$ for $t \in (-\infty, T]$ such that

$$\|\boldsymbol{u}\|_{\mathfrak{L}_T} = (\mathcal{E} \sup_{t \in (-\infty, T]} \|\boldsymbol{u}(t,\boldsymbol{x})\|^2)^{\frac{1}{2}} < +\infty.$$

For any $\boldsymbol{u} \in \mathfrak{L}_T$, let

$$\begin{aligned}\Gamma(\boldsymbol{u})(t) = U(t,t_0)\boldsymbol{\varphi}(t_0) + \int_{t_0}^{t} U(t,s)\boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u}))ds \\ + \int_{t_0}^{t} U(t,s)\boldsymbol{G}(s,\mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(s,\boldsymbol{x}),\end{aligned} \quad (6)$$

for $t \in [t_0, T]$ and $\Gamma(\boldsymbol{u})(t) = \boldsymbol{\varphi}(t)$ for $t \in (-\infty, t_0]$.
First, we will deduce that Γ maps $\mathfrak{L}_T$ into $\mathfrak{L}_T$. Indeed, by the Jensen inequality, we have

$$\begin{aligned}\mathcal{E}\sup_{t_0\le t\le T}\|\Gamma(\boldsymbol{u})(t)\|^2 &\le 3\mathcal{E}\sup_{t_0\le t\le T}\|U(t,t_0)\boldsymbol{\varphi}(t_0)\|^2 \\ &+3\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)\boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u}))ds\|^2 \\ &+3\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)\boldsymbol{G}(s,\mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}\|^2.\end{aligned} \quad (7)$$

Obviously, $\mathcal{E}\sup_{t_0\le t\le T}\|U(t,t_0)\boldsymbol{\varphi}(t_0)\|^2 \le N^2(T)\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2$. It follows from H3 that

$$\int_{t_0}^{t}\|\boldsymbol{f}(s,\boldsymbol{0})\|^2 ds \vee \int_{t_0}^{t}\|\boldsymbol{G}(s,\boldsymbol{0})\|^2 ds \le M(T), \quad (8)$$

where $M(T) = m^2(T)(T-t_0)$. By the Hölder's inequality, H2 and (8), we obtain

$$\begin{aligned}&\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)\boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u}))ds\|^2 \\ &\le 2N^2(T)(T-t_0) \\ &\quad \mathcal{E}\sup_{t_0\le t\le T}[L^2\int_{t_0}^{t}\|\int_{-\infty}^{0} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta,\boldsymbol{x})\|^2 ds + M(T)] \\ &\le 2N^2(T)(T-t_0)[nL^2\|\hat{\boldsymbol{\eta}}\|_F^2(T-t_0)\|\boldsymbol{u}\|_{\mathfrak{L}_T}^2 + M(T)].\end{aligned} \quad (9)$$

By the Itô isometry [20], H2 and (8), we find that

$$\begin{aligned}&\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)\boldsymbol{G}(s,\mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(s,\boldsymbol{x})\|^2 \\ &\le N^2(T)\mathcal{E}\sup_{t_0\le t\le T}\int_{t_0}^{t}\|\boldsymbol{G}(s,\mathcal{S}(\boldsymbol{u}))\|^2 ds \\ &\le 2N^2(T)\mathcal{E}\sup_{t_0\le t\le T}(L^2\int_{t_0}^{t}\|\int_{-\infty}^{0} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t,\boldsymbol{x})\|^2 ds \\ &\quad + \int_{t_0}^{t}\|\boldsymbol{G}(s,\boldsymbol{0})\|^2 ds) \\ &\le 2N^2(T)[nL^2\|\hat{\boldsymbol{\eta}}\|_F^2(T-t_0)\|\boldsymbol{u}\|_{\mathfrak{L}_T}^2 + M(T)].\end{aligned} \quad (10)$$

Hence, equations (6)-(10) imply that $\|\Gamma(\boldsymbol{u})(t)\|_{\mathfrak{L}_T}^2 < +\infty$ if $\boldsymbol{u} \in \mathfrak{L}_T$, indicating that Γ maps $\mathfrak{L}_T$ into $\mathfrak{L}_T$.

Then, we will deduce that Γ is a strict contraction map on $\mathfrak{L}_T$. For any $\tilde{\boldsymbol{u}}, \bar{\boldsymbol{u}} \in \mathfrak{L}_T$, we have

$$\begin{aligned}&\mathcal{E}\sup_{t_0\le t\le T}\|\Gamma(\tilde{\boldsymbol{u}}) - \Gamma(\bar{\boldsymbol{u}})\|^2 \\ &\le 2\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)[\boldsymbol{f}(s,\mathcal{S}(\tilde{\boldsymbol{u}})) - \boldsymbol{f}(s,\mathcal{S}(\bar{\boldsymbol{u}}))]ds\|^2 \\ &\quad +2\mathcal{E}\sup_{t_0\le t\le T}\|\int_{t_0}^{t} U(t,s)[\boldsymbol{G}(s,\mathcal{S}(\tilde{\boldsymbol{u}})) - \boldsymbol{G}(s,\mathcal{S}(\bar{\boldsymbol{u}}))d\boldsymbol{W}]\|^2 \\ &\le 2N^2(T)(T-t_0)\mathcal{E}\sup_{t_0\le t\le T}\int_{t_0}^{t}\|\boldsymbol{f}(s,\mathcal{S}(\tilde{\boldsymbol{u}})) - \boldsymbol{f}(s,\mathcal{S}(\bar{\boldsymbol{u}}))\|^2 ds \\ &\quad +2N^2(T)\mathcal{E}\sup_{t_0\le t\le T}\int_{t_0}^{t}\|\boldsymbol{G}(s,\mathcal{S}(\tilde{\boldsymbol{u}})) - \boldsymbol{G}(s,\mathcal{S}(\bar{\boldsymbol{u}}))\|^2 ds \\ &\le 2N^2(T)(T-t_0+1)L^2 \\ &\quad \mathcal{E}\sup_{t_0\le t\le T}\int_{t_0}^{t}\|\int_{-\infty}^{0} d\boldsymbol{\eta}(\theta)[\tilde{\boldsymbol{u}}(t+\theta,\boldsymbol{x}) - \bar{\boldsymbol{u}}(t+\theta,\boldsymbol{x})]\|^2 ds \\ &\le 2nL^2\|\hat{\boldsymbol{\eta}}\|_F^2 N^2(T)(T-t_0)(T-t_0+1)\|\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\mathfrak{L}_T}^2.\end{aligned} \quad (11)$$

Hence Γ is contractive by (5). Consequently, Γ has a unique fixed point $\boldsymbol{u}$ in $\mathfrak{L}_T$, which is the unique mild solution to (1). □

Proposition 2. Suppose that H1-H3 hold. Then the mild solution to (1) satisfies the prior estimate $\mathcal{E}\|\boldsymbol{u}\|_{\mathfrak{L}_T}^2 \le B(T)$, $t \in [t_0, T]$, where $B(T)$ is a positive constant.

Proof. Considering the mild solution $\boldsymbol{u}$ to equation (1), we construct a Lyapunov-Krasovskii functional $V(t) = e^{\lambda(t-t_0)}\|\boldsymbol{u}\|^2$ where $\lambda \ge 2(\alpha\kappa^2 + \beta)$. Thanks to the Itô formula, we have

$$dV(t) = \mathcal{L}V(t)dt + 2e^{\lambda(t-t_0)}(\boldsymbol{u}, \boldsymbol{G}d\boldsymbol{W}), \quad (12)$$

where the differential operator $\mathcal{L}V(t)$ is given by

$$\begin{aligned}\mathcal{L}V(t) =& \lambda V(t) + 2e^{\lambda(t-t_0)}(\boldsymbol{u}, \mathcal{A}(t)\boldsymbol{u}) \quad (13) \\ &+ 2e^{\lambda(t-t_0)}(\boldsymbol{u}, \boldsymbol{f}(t,\mathcal{S}(\boldsymbol{u}))) + e^{\lambda(t-t_0)}\mathrm{tr}(\boldsymbol{G}Q\boldsymbol{G}^*).\end{aligned}$$

Integrating (12) and taking expectation, we have

$$\begin{aligned}\mathcal{E}V(t) = \mathcal{E}\|\boldsymbol{\varphi}(0)\|^2 + \lambda\int_{t_0}^{t}\mathcal{E}V(s)ds \\ +2\int_{t_0}^{t} e^{\lambda(s-t_0)}\mathcal{E}(\boldsymbol{u}, \mathcal{A}(s)\boldsymbol{u})ds \\ +2\int_{t_0}^{t} e^{\lambda(s-t_0)}\mathcal{E}(\boldsymbol{u}, \boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u})))ds \\ +\int_{t_0}^{t} e^{\lambda(s-t_0)}\mathcal{E}\mathrm{tr}(\boldsymbol{G}Q\boldsymbol{G}^*)ds.\end{aligned} \quad (14)$$

By employing the Gaussian theorem, Dirchlet boundary condition and Poincaré inequallity, we obtain

$$\begin{aligned}&\mathcal{E}\sum_{i=1}^{n}\int_\Omega u_i\sum_{j=1}^{n}\frac{\partial(D_{ij}(s,\boldsymbol{x})\frac{\partial u_i}{\partial x_j})}{\partial x_j}d\boldsymbol{x} \\ &\le -\mathcal{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega D_{ij}(s,\boldsymbol{x})(\frac{\partial u_i}{\partial x_j})^2 d\boldsymbol{x} \le -\alpha\kappa^2\mathcal{E}\|\boldsymbol{u}(s,\boldsymbol{x})\|^2.\end{aligned}$$

Therefore, $\mathcal{E}(\boldsymbol{u}, \mathcal{A}(s)\boldsymbol{u}) \le (-\alpha\kappa^2 - \beta)\mathcal{E}\|\boldsymbol{u}(s,\boldsymbol{x})\|^2$. Applying Young inequality, H2 and H3 results in

$$\begin{aligned}2\mathcal{E}(\boldsymbol{u}, \boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u}))) &\le \mathcal{E}\|\boldsymbol{u}(s,\boldsymbol{x})\|^2 + \mathcal{E}\|\boldsymbol{f}(s,\mathcal{S}(\boldsymbol{u}))\|^2 \\ &\le \mathcal{E}\|\boldsymbol{u}(s,\boldsymbol{x})\|^2 + 2L^2\mathcal{E}\|\mathcal{S}(\boldsymbol{u})\|^2 + 2m^2(T),\end{aligned} \quad (15)$$

and

$$\mathcal{E}\mathrm{tr}(\boldsymbol{G}(t,\mathcal{S}(\boldsymbol{u}))Q\boldsymbol{G}(t,\mathcal{S}(\boldsymbol{u}))^*) \le 2L^2\mathcal{E}\|\mathcal{S}(\boldsymbol{u})\|^2 + 2m^2(T). \quad (16)$$

Since $\int_{-\infty}^{0} d\boldsymbol{\eta}(\theta) = \hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}(\theta)$ is non-decreasing bounded variation function, we have

$$\begin{aligned}&\int_{t_0}^{t} e^{\lambda(s-t_0)}\mathcal{E}\|\int_{-\infty}^{0} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(s+\theta,\boldsymbol{x})\|^2 ds \\ &\le 2\int_{t_0}^{t} e^{\lambda(s-t_0)}\mathcal{E}(\|\int_{-\infty}^{t_0-s} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(s+\theta,\boldsymbol{x})\|^2 \\ &\quad +\|\int_{t_0-s}^{0} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(s+\theta,\boldsymbol{x})\|^2)ds \\ &\le 2nc_1(T)\|\hat{\boldsymbol{\eta}}\|_F^2\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2 \\ &\quad +2\mathcal{E}\int_{t_0}^{t} e^{\lambda(s-t_0)}\|\int_{t_0}^{s} d\boldsymbol{\eta}(\zeta - s)\boldsymbol{u}(\zeta,\boldsymbol{x})\|^2 ds \\ &\le 2nc_1(T)\|\hat{\boldsymbol{\eta}}\|_F^2\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2 \\ &\quad +2\|\hat{\boldsymbol{\eta}}\|_F^2 mes(\mathbb{O})\int_{t_0}^{t} e^{\lambda(s-t_0)}\int_{t_0}^{s}\mathcal{E}\|\boldsymbol{u}(\zeta,\boldsymbol{x})\|^2 d\zeta ds \\ &\le 2c_1(T)\|\hat{\boldsymbol{\eta}}\|_F^2 \\ &\quad (n\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2 + mes(\mathbb{O})\int_{t_0}^{t} e^{\lambda(\zeta-t_0)}\mathcal{E}\|\boldsymbol{u}(\zeta,\boldsymbol{x})\|^2 d\zeta),\end{aligned} \quad (17)$$

where $c_1(T) = (e^{\lambda(T-t_0)} - 1)/\lambda$ and $mes(\mathbb{O})$ stands for the Lebesgue measure of the domain $\mathbb{O}$. Thus, we can infer from (12)-(17) that $\mathcal{E}V(t) \le c_2(T)\int_{t_0}^{t}\mathcal{E}V(s)ds + c_3(T)$, where $c_2(T) = \lambda - 2\alpha\kappa^2 - 2\beta + 1 + 8L^2c_1(T)\|\hat{\boldsymbol{\eta}}\|_F^2 mes(\mathbb{O})$ and $c_3(T) = 4c_1(T)m^2(T) + (1 + 8nL^2c_1(T)\|\hat{\boldsymbol{\eta}}\|_F^2)\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2$. From Gronwall inequality, we get $\mathcal{E}V(t) \le c_3(T)e^{c_2(T)(t-t_0)}$ which implies that $\mathcal{E}\|\boldsymbol{u}(t,\boldsymbol{x})\|^2 \le c_3(T)e^{(c_2(T)-\lambda)(t-t_0)}$. Then, the proposition is proved. □

Remark 1. From Proposition 2, we can see that the mild solution cannot blow up in finite time where the assumption H3 is weaker than the linear growth condition.

From Propositions 1 and 2, the local existence-uniqueness and boundedness of the mild solution are obtained. Then the following result is obvious.

Theorem 1. Suppose that H1-H3 hold. Given $\boldsymbol{\varphi} \in \mathfrak{C}^b_{\mathcal{F}_0}$, then there exists a unique global mild solution $\boldsymbol{u}$ to equation (1) and $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}, \omega) \in L^2(\mathbb{R} \times \mathbb{O} \times \Omega, \mathbb{R}^n)$.

Proof. From Propositions 1 and 2, we have the existence and boundedness of the mild solution of equation (1) in $[t_0, T]$. Then $\boldsymbol{u}(t)$ can be extended to $[t_0, b]$ for any $b > 0$. Since $\boldsymbol{u}(t)$ can be defined on every finite subinterval $[t_0, b]$ of $[t_0, \infty)$, we can obtain that equation (1) has a unique mild solution $\boldsymbol{u}(t)$ on $[t_0, \infty)$ [23]. □

Remark 2. In [24], an existence-uniqueness theorem for stochastic functional differential equations with finite delay is established under the condition: $\int_{t_0}^t \|\boldsymbol{f}(s, \boldsymbol{0})\|^2 ds \vee \int_{t_0}^t \|\boldsymbol{G}(s, \boldsymbol{0})\|^2 ds \leq +\infty$. This means that the blow-up of mild solution originates from time variable t in $\boldsymbol{f}(t, \boldsymbol{u})$ and $\boldsymbol{G}(t, \boldsymbol{u})$. However, the integral condition is inconvenient to be verified, whereas condition H3 not only ensures the boundedness of mild solution in finite time but also is convenient to be verified.

Furthermore, the existence-uniqueness of the mild solution can be obtained, if the following spatiotemporal Lipschitz condition holds.

H4: $\boldsymbol{f}$ and $\boldsymbol{G}$ satisfy the spatiotemporal Lipschitz condition: $\|\boldsymbol{f}(t, \boldsymbol{u}) - \boldsymbol{f}(t, \boldsymbol{v})\| \vee \|\boldsymbol{G}(t, \boldsymbol{u}) - \boldsymbol{G}(t, \boldsymbol{v})\| \vee \|\boldsymbol{G}(t, \boldsymbol{u}) - \boldsymbol{G}(t, \boldsymbol{v})\|_* \leq L\|\boldsymbol{u} - \boldsymbol{v}\|$, and $\|\boldsymbol{f}(t, \boldsymbol{0}) - \boldsymbol{f}(s, \boldsymbol{0})\| \vee \|\boldsymbol{G}(t, \boldsymbol{0}) - \boldsymbol{G}(s, \boldsymbol{0})\| \vee \|\boldsymbol{G}(t, \boldsymbol{0}) - \boldsymbol{G}(s, \boldsymbol{0})\|_* \leq L|t - s|$, where $t, s \geq t_0$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{L}$.

Corollary 1. Suppose that H1 and H4 hold. Given $\boldsymbol{\varphi} \in \mathfrak{C}^b_{\mathcal{F}_0}$, then there exists a unique global mild solution $\boldsymbol{u}$ to equation (1) and $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}, \omega) \in L^2(\mathbb{R} \times \mathbb{O} \times \Omega, \mathbb{R}^n)$.

IV. Global Exponential Stability

Our goal in this section is to investigate global exponential stability of nonautonomous stochastic functional differential system (1). Let us make a further assumption:

H5: $\boldsymbol{f}(t, \boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{G}(t, \boldsymbol{0}) = \boldsymbol{0}$, for $t \in [t_0, +\infty)$.

As a result, equation (1) admits a trivial solution $\boldsymbol{u}(t, \boldsymbol{x}) \equiv 0$ which is the equilibrium point.

Theorem 2. Assume that H1, H2 and H5 hold. Then the trivial solution to (1) is globally exponentially stable in the mean-square sense, if $-2\alpha\kappa^2 - 2\beta + 1 + 4nL^2\|\hat{\boldsymbol{\eta}}\|_F^2 < 0$.

Proof. Obviously, since H5 implies H3, a mild solution to (1) exists under H1, H2 and H5. Denote the mild solution by $\boldsymbol{u}(t, \boldsymbol{x})$ and consider the following Lyapunov-Krasovskii functional

$$V(t) = \|\boldsymbol{u}(t, \boldsymbol{x})\|^2. \tag{18}$$

Thanks to the Itô formula, we have $dV(t) = \mathcal{L}V dt + (\boldsymbol{u}, \boldsymbol{G}d\boldsymbol{W})$, where the differential operator is given by

$$\mathcal{L}V(t) = 2(\boldsymbol{u}, \mathcal{A}(t)\boldsymbol{u}) + 2(\boldsymbol{u}, \boldsymbol{f}(t, \mathcal{S}(\boldsymbol{u}))) + \mathrm{tr}(\boldsymbol{G}Q\boldsymbol{G}^*). \tag{19}$$

Taking expectation of $dV(t)$, we know that $d\mathcal{E}V(t) = \mathcal{E}\mathcal{L}V dt$. From (15)-(16), H2 and H5, we obtain

$$2\mathcal{E}(\boldsymbol{u}, \mathcal{A}(t)\boldsymbol{u}) \leq -2(\alpha\kappa^2 + \beta)\mathcal{E}V(t), \tag{20}$$

$$2\mathcal{E}(\boldsymbol{u}, \boldsymbol{f}(t, \mathcal{S}(\boldsymbol{u}))) \leq \mathcal{E}V(t) + L^2\mathcal{E}\|\mathcal{S}(\boldsymbol{u})\|^2, \tag{21}$$

and

$$\mathcal{E}\mathrm{tr}(\boldsymbol{G}Q\boldsymbol{G}^*) \leq L^2\mathcal{E}\|\mathcal{S}(\boldsymbol{u})\|^2. \tag{22}$$

Since $\int_{-\infty}^0 d\boldsymbol{\eta}(\theta) = (\hat{\eta}_{ij})_{n\times n}$, $\hat{\eta}_{ij} > 0$ and $\boldsymbol{\eta}(\theta)$ is non-decreasing bounded variation function, there exists a constant $\tau \geq t - t_0$ such that $\int_{-\infty}^{-\tau} d\boldsymbol{\eta}(\theta) \leq \epsilon$ for any $\epsilon \geq 0$. So,

$$\begin{aligned}&\mathcal{E}\|\mathcal{S}(\boldsymbol{u})\|^2 = \mathcal{E}\|\textstyle\int_{-\infty}^0 d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta, \boldsymbol{x})\|^2\\ &\leq 2\mathcal{E}(\|\textstyle\int_{-\infty}^{-\tau} d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta, \boldsymbol{x})\|^2 + \|\int_{-\tau}^0 d\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta, \boldsymbol{x})\|^2)\\ &\leq 2n(\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2\epsilon^2 + \|\hat{\boldsymbol{\eta}}\|_F^2 \sup_{-\tau\leq\theta\leq 0} \mathcal{E}\|\boldsymbol{u}(t+\theta, \boldsymbol{x})\|^2).\end{aligned} \tag{23}$$

Combining (18)-(23), we have

$$\frac{d\mathcal{E}V(t)}{dt} \leq -k_1\mathcal{E}V(t) + k_2 \sup_{-\tau\leq\theta\leq 0} \mathcal{E}V(t+\theta) + k_3\epsilon^2, \tag{24}$$

where $k_1 = 2\alpha\kappa^2 + 2\beta - 1$, $k_2 = 4nL^2\|\hat{\boldsymbol{\eta}}\|_F^2$ and $k_3 = 4nL^2\mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2$. Then, we consider the function $\sigma(\lambda) = -\lambda + k_1 - k_2e^{\lambda\tau}$, where $\lambda \in [0, +\infty)$. Since $k_1 - k_2 = 2\alpha\kappa^2 + 2\beta - 1 - 4nL^2\|\hat{\boldsymbol{\eta}}\|_F^2 > 0$ and $k_2 > 0$, we obtain $\sigma(0) > 0$ and $\sigma(\lambda)$ is continuous and monotonous. Furthermore, $\sigma(\lambda) \to -\infty$ as $\lambda \to +\infty$. Thus, there exists a constant $\lambda_0 > 0$ such that $-\lambda_0 + k_1 - k_2e^{\lambda_0\tau} > 0$. Let $\Phi(t, x, y) = -k_1x + k_2y + k_3\epsilon^2$ and $\varsigma(t) = Me^{-\lambda_0(t-t_0)} + \frac{k_3\epsilon^2}{k_1-k_2}$, where $M = \mathcal{E}\|\boldsymbol{\varphi}\|_{\mathfrak{C}}^2$. Obviously, $\mathcal{E}V(t) \leq M < \varsigma(t)$ for $t \leq t_0$. For $t \geq t_0$, we have

$$\begin{aligned}D^+\varsigma(t) &= -\lambda_0Me^{-\lambda_0(t-t_0)} > (-k_1 + k_2e^{\lambda_0\tau})Me^{-\lambda_0(t-t_0)}\\ &\geq -k_1\varsigma(t) + k_2 \sup_{-\tau\leq\theta\leq 0} \varsigma(t+\theta) + k_3\epsilon^2 = \Phi(t, \varsigma(t), \bar{\varsigma}(t)),\end{aligned} \tag{25}$$

and $D^+\mathcal{E}V(t) \leq \Phi(t, \mathcal{E}V(t), \mathcal{E}\bar{V}(t))$. Based on Lemma 2, we have $\mathcal{E}\|\boldsymbol{u}(t, \boldsymbol{x})\|^2 \leq Me^{-\lambda_0(t-t_0)} + k_3\epsilon^2$. As $\epsilon \to 0$, we obtain the global exponential stability of system (1) in the mean-square sense. □

When $D_{ij}(t, \boldsymbol{x}) = \tilde{D}_{ij}(\boldsymbol{x}) \geq \alpha > 0$ and $c_i(t, \boldsymbol{x}) = 0$ $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, n)$, the reaction-diffusion linear operator is a generator of a strongly continuous semigroup $\tilde{U}(t)$. Then we have the following results for stochastic functional differential system with S-type distributed delay and an autonomous reaction-diffusion term

$$\begin{aligned}d\boldsymbol{u}(t, \boldsymbol{x}) = [\nabla\cdot(\boldsymbol{D}(\boldsymbol{x}) \circ \nabla\boldsymbol{u}(t, \boldsymbol{x})) + \boldsymbol{f}(t, \mathcal{S}(\boldsymbol{u}))]dt\\ +\boldsymbol{G}(t, \mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(t, \boldsymbol{x}).\end{aligned} \tag{26}$$

Corollary 2. Assume that H2 and H5 hold. Then the system (26) is globally exponentially stable in the mean-square sense, if $-2\alpha\kappa^2 + 1 + 4nL^2\|\hat{\boldsymbol{\eta}}\|_F^2 < 0$.

Furthermore, if we consider the following stochastic functional differential system with finite delay and an autonomous reaction-diffusion term

$$\begin{aligned}d\boldsymbol{u}(t, \boldsymbol{x}) = [\nabla\cdot(\boldsymbol{D}(\boldsymbol{x}) \circ \nabla\boldsymbol{u}(t, \boldsymbol{x})) + \boldsymbol{f}(t, \boldsymbol{u}_t)]dt\\ +\boldsymbol{G}(t, \boldsymbol{u}_t)d\boldsymbol{W}(t, \boldsymbol{x}),\end{aligned} \tag{27}$$

where $\boldsymbol{u}_t = \boldsymbol{u}(t + r, \boldsymbol{x})$ and $-\tau \leq r \leq 0$, we have the following result.

Corollary 3. Assume that H2 and H5 hold. Then the system (27) is almost surely exponentially stable, if $-2\alpha\kappa^2 + 1 + 2L^2 < 0$.

Proof. Based on the argument of Theorem 2 and Halanay inequality, it is straightforward to deduce that the mild solution to (27) is globally exponentially stable in the mean-square sense, if H2 and H5 hold. Denote the mild solution by $\boldsymbol{u}(t, \boldsymbol{x})$. By Definition 1 and the Jensen inequality, we have

$$\begin{aligned}&\mathcal{E} \sup_{p\le t\le p+1} \|\boldsymbol{u}(t,\boldsymbol{x})\|^2\\ &\le 3 \sup_{p\le t\le p+1} \mathcal{E}\|\tilde{U}(t-p+1)\boldsymbol{u}(p-1,\boldsymbol{x})\|^2\\ &+3 \sup_{p\le t\le p+1} \mathcal{E}\|\textstyle\int_{p-1}^t \tilde{U}(t-s)\boldsymbol{f}(s,\boldsymbol{u}_s)ds\|^2\\ &+3 \sup_{p\le t\le p+1} \mathcal{E}\|\textstyle\int_{p-1}^t \tilde{U}(t-s)\boldsymbol{G}(s,\boldsymbol{u}_s)d\boldsymbol{W}\|^2.\end{aligned} \tag{28}$$

Since $\tilde{U}(t)$ is a contraction map [16] and the mild solution is globally exponentially stable, we can infer from the Hölder inequality and the Burkholder-Davis-Gundy inequality [20] that there exists a positive constant c such that

$$\sup_{p\le t\le p+1} \mathcal{E}\|\tilde{U}(t-p+1)\boldsymbol{u}(p-1,\boldsymbol{x})\|^2 \le k_4 e^{-\lambda_0(p-1-t_0)}, \tag{29}$$

$$\begin{aligned}&\sup_{p\le t\le p+1} \mathcal{E}\|\textstyle\int_{p-1}^t \tilde{U}(t-p+1)\boldsymbol{f}(s,\boldsymbol{u}_s)ds\|^2\\ &\le 2L^2 \textstyle\int_{p-1}^{p+1} \mathcal{E}\|\boldsymbol{u}_s\|^2 ds \le 4k_4L^2e^{\lambda_0\tau}e^{-\lambda_0(p-1-t_0)},\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\sup_{p\le t\le p+1} \mathcal{E}\|\textstyle\int_{p-1}^t \tilde{U}(t-s)\boldsymbol{G}(s,\boldsymbol{u}_s)d\boldsymbol{W}\|^2\\ &\le 2ck_4L^2e^{\lambda_0\tau}e^{-\lambda_0(p-1-t_0)}.\end{aligned} \tag{31}$$

where $k_4 = \mathcal{E}\|\boldsymbol{\varphi}\|^2_{\mathfrak{C}}$. Thus, $\mathcal{E}\sup_{p\le t\le p+1}\|\boldsymbol{u}(t,\boldsymbol{x})\|^2 \le k_5e^{\lambda_0\tau}e^{-\lambda_0(p-1-t_0)}$, where $k_5 = 3k_4(1 + 4L^2e^{\lambda_0\tau} + 2cL^2e^{\lambda_0\tau})$. Based on the Chebyshev inequality and Borel-Cantelli lemma, we can conclude that the system (27) is almost surely exponentially stable. □

Remark 3. In [1], [15], [16], Liang and Wang *et al.* investigate the well-posedness and stability of autonomous SRDNNs, while the reaction-diffusion term considered here is nonautonomous and the obtained results can be applied to nonautonomous SRDNNs (see Example 3 later). Therefore, our results are more general.

Remark 4. In most existing works about SRDNNs [13], [14], [18], [19], researchers have investigated the stability and synchronization. But few authors have considered the existence of the solutions, which is an indispensable step before stability analysis. The results discussed here can provide an existence-uniqueness theorem of the solution for those works.

Remark 5. The discussed system includes the S-type distributed delay, which is ignored in recent works about SRDNNs [1], [15], [18], [19]. If the infinite delayed system degenerates into the finite delayed system (26), we show that it is almost surely exponentially stable from Corollary 3, which is further result of exponential stability for systems in [1], [15], [19]. In [18], the authors obtained excellent results about SRDNNs with continuous distributed delays, but the stability conditions are irrelevant to the reaction-diffusion term and distributed delays. The stability criteria obtained here indicate that they contribute to the stability. Therefore, the obtained results are more effective.

V. Neural network models

In this section, we present some results about SRDNNs and an illustrative example.

Example 1. Consider the following nonautonomous stochastic reaction-diffusion Hopfield neural networks with S-type distributed delays

$$\begin{aligned}d\boldsymbol{u}(t,\boldsymbol{x}) = [&\nabla\cdot(\boldsymbol{D}(t,\boldsymbol{x})\circ\nabla\boldsymbol{u}(t,\boldsymbol{x})) - \boldsymbol{C}(t)\boldsymbol{u}(t,\boldsymbol{x})\\ &+\boldsymbol{B}_1(t)\tilde{\boldsymbol{f}}(\boldsymbol{u}) + \boldsymbol{B}_2(t)\tilde{\boldsymbol{f}}(\mathcal{S}(\boldsymbol{u})) + \boldsymbol{I}]dt\\ &+\boldsymbol{E}(t)\tilde{\boldsymbol{G}}(\mathcal{S}(\boldsymbol{u}))d\boldsymbol{W}(t,\boldsymbol{x}),\end{aligned} \tag{32}$$

where $\boldsymbol{C}(t) = diag(c_1(t), c_2(t), \cdots, c_n(t))$ and $D_{ij}(t,\boldsymbol{x})$. $c_i(t)$ are real-valued smooth functions with bounded derivatives, $D_{ij}(t,\boldsymbol{x}) \ge \alpha > 0$, $c_i(t) \ge \beta$ and $\beta > \alpha/2$. $\boldsymbol{B}_1(t) = (B_{1ij}(t))_{n\times n}$, $\boldsymbol{B}_2(t) = (B_{2ij}(t))_{n\times n}$, $\boldsymbol{E}(t) = (E_{ij}(t))_{n\times n}$, $\boldsymbol{I} = (I_1, I_2, \cdots, I_n)^T$, $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_n)^T$ and $\tilde{\boldsymbol{G}} = (\tilde{G}_{ij})_{n\times m} \in M_2^{n,m}(0,t)$.

Assertion 1. Assume that

H6: $\|\tilde{\boldsymbol{f}}(\boldsymbol{u}) - \tilde{\boldsymbol{f}}(\boldsymbol{v})\| \vee \|\tilde{\boldsymbol{G}}(\boldsymbol{u}) - \tilde{\boldsymbol{G}}(\boldsymbol{v})\| \vee \|\tilde{\boldsymbol{G}}(\boldsymbol{u}) - \tilde{\boldsymbol{G}}(\boldsymbol{v})\|_* \le \tilde{L}\|\boldsymbol{u} - \boldsymbol{v}\|$ and $\tilde{\boldsymbol{f}}(\boldsymbol{0}) = \tilde{\boldsymbol{G}}(\boldsymbol{0}) = \boldsymbol{0}$, where $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{L}$;

H7: $\boldsymbol{B}_1(t)$, $\boldsymbol{B}_2(t)$ and $\boldsymbol{E}(t)$ are bounded and continuous functions in $[t_0, +\infty)$.

Then the system (32) has a unique global mild solution. Moreover, the system (32) is globally exponentially stable in the mean-square sense, if $-2\alpha\kappa^2 - 2\beta + 1 + 4n\tilde{L}^2\Upsilon\|\boldsymbol{\eta}\|_F^2 < 0$, where $\Upsilon = \sup_t\{\|\boldsymbol{B}_1(t)\|_F^2, \|\boldsymbol{B}_2(t)\|_F^2, \|\boldsymbol{E}(t)\|_F^2\}$.

Example 2. Consider the following nonautonomous stochastic reaction-diffusion Cohen-Grossberg neural network

$$\begin{aligned}d\boldsymbol{u}(t,\boldsymbol{x}) = [&\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) - \bar{\boldsymbol{C}}(t,\boldsymbol{u})(\bar{\boldsymbol{B}}(t,\boldsymbol{u})\\ &-\bar{\boldsymbol{f}}(t,\boldsymbol{u}_t) - \boldsymbol{I})]dt + \bar{\boldsymbol{G}}(t,\boldsymbol{u}_t)d\boldsymbol{W}(t,\boldsymbol{x}),\end{aligned} \tag{33}$$

where $D_{ij} \ge \alpha > 0$, $\bar{\boldsymbol{C}} = \text{diag}(\bar{c}_1, \cdots, \bar{c}_n)$, $\bar{\boldsymbol{B}} = (\bar{b}_1, \cdots, \bar{b}_n)^T$, $\bar{\boldsymbol{f}} = (\bar{f}_1, \cdots, \bar{f}_n)^T$ and $\bar{\boldsymbol{G}} = (\bar{G}_{ij})_{n\times m} \in M_2^{n,m}(0,t)$.

Assertion 2. Assume that

H8: $\|\bar{\boldsymbol{C}}(t,\boldsymbol{u}) - \bar{\boldsymbol{C}}(t,\boldsymbol{v})\| \vee \|\bar{\boldsymbol{B}}(t,\boldsymbol{u}) - \bar{\boldsymbol{B}}(t,\boldsymbol{v})\| \vee \|\bar{\boldsymbol{f}}(t,\boldsymbol{u}) - \bar{\boldsymbol{f}}(t,\boldsymbol{v})\| \vee \|\bar{\boldsymbol{G}}(t,\boldsymbol{u}) - \bar{\boldsymbol{G}}(t,\boldsymbol{v})\| \vee \|\bar{\boldsymbol{G}}(t,\boldsymbol{u}) - \bar{\boldsymbol{G}}(t,\boldsymbol{v})\|_* \le \bar{L}\|\boldsymbol{u} - \boldsymbol{v}\|$ and $\bar{\boldsymbol{C}}(t,\boldsymbol{0}) = \bar{\boldsymbol{G}}(t,\boldsymbol{0}) = \boldsymbol{0}$ where $t \ge t_0$, $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{L}$;

H9: $0 < m \le \bar{c}_i(t,\mu) \le M$, $\bar{b}_i(t,\mu)/\mu \ge \beta$, $i = 1, 2, \cdots, n$, $\mu \in \mathbb{R}$.

Then the system (33) has a unique global mild solution. Moreover, the system (33) is almost surely exponentially stable, if $-2\alpha\kappa^2 - 2m\beta + 1 + 2\bar{L}^2M^2 < 0$.

Remark 6. The discussed systems include a nonautonomous reaction-diffusion term and an infinite dimensional Wiener processes, so the obtained results also extend the results about nonautonomous neural networks [4], [17], [25] and stochastic neural networks [13], [15], [26].

Example 3. Consider the following 1-D nonautonomous SRDNN with delay

$$du(t,x) = [D(t)\Delta u(t,x) - 2.5u(t,x) + S(u)]dt + S(u)dw(t), \tag{34}$$

with the homogeneous Dirichlet boundary condition and the initial condition $u(t,x) = \frac{1}{5}x\cos((\frac{1}{3}t+\frac{1}{2})\pi)$ for $t \in [-3, 0]$. $D(t) = 2+\sin(t)$, $S(u) = \int_{-\infty}^{0} k(r)u_t(r)dr$, $\mathbb{O} = [-5, 5]$ and $w(t)$ is 1-D standard Brownian motion. Obviously, $\alpha = 1$, $\beta = 2.5$, $L = 1$, and $\int_{-\infty}^{0} e^r dr = 1$. Then it follows from Theorem 2 that the system (34) is globally exponentially stable in the mean-square sense. Fig. 1(a) depicts the dynamical behavior of this SRDNN which is shown to be stable.

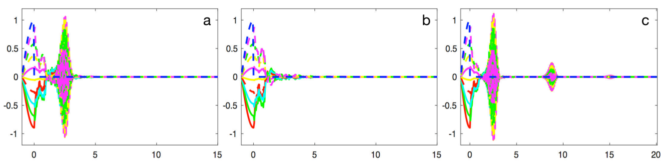

Fig. 1. Time trajectories of systems (34) with different reaction-diffusion terms and delays. (a) $D(t) = 2 + \sin(t)$, $k(r) = e^r$. (b) $D(t) = 1$, $k(r) = e^r$. (c) $D(t) = 2 + \sin(t)$, $k(r) = \frac{1}{N}\sum_{n=1}^{N} e^r\delta(r + n)$.

Remark 7. To show the effect of different reaction-diffusion terms and delays, Fig. 1(b) depicts trajectory of system (34) with $D(t) = 1$ which equals to the minimum of the nonautonomous reaction-diffusion term $D(t) = 2+\sin(t)$. Then, the inequality for stability in Theorem 2 is also satisfied with same α. By comparison, the nonautonomous system fluctuates more heavily than the autonomous one, indicating that the nonautonomous reaction-diffusion term causes a significant change of SRDNN. Fig. 1(c) illustrates system (34) with S-type delay $k(r) = \frac{1}{N}\sum_{n=1}^{N} e^r\delta(r + n)$ where $N = [t + 3]$. Compared with Fig. 1(a) which includes general unbounded distributed time delay, the discontinuity of the delay kernel also influences SRDNN and distinguishes the S-type distributed delay from others. Thus, the obtained results are more general than those systems in [1], [8], [10], [14], [15], [18].

VI. Conclusion

In this brief, we study a nonautonomous stochastic reaction-diffusion functional differential system with S-type distributed delay. First, existence-uniqueness of mild solution for this system is established based on evolution system theory under Lipschitz condition. Then, asymptotic behavior of the solution is discussed and some criteria of global exponential stability are obtained by the Lyapunov method and truncation method. Finally, the applicability of the proposed theories is verified by some neural network models and an illustrative example.

Acknowledgment

The authors would like to thank the editors and reviewers for their constructive comments and suggestions.

References

[1] X. Liang, L. Wang, Y. Wang, and R. Wang, "Dynamical behavior of delayed reaction-diffusion hopfield neural networks driven by infinite dimensional wiener processes," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 9, pp. 1816–1826, 2016.

[2] L. V. Hien, T. T. Loan, B. T. H. Trang, and H. Trinh, "Existence and global asymptotic stability of positive periodic solution of delayed cohen-grossberg neural networks," *Appl. Math. Comput.*, vol. 240, pp. 200–212, 2014.

[3] W.-H. Chen, S. Luo, and W. X. Zheng, "Generating globally stable periodic solutions of delayed neural networks with periodic coefficients via impulsive control," *IEEE Trans. Cybern.*, vol. 47, no. 7, pp. 1590–1603, 2017.

[4] S. Esteves, E. Gökmen, and J. J. Oliveira, "Global exponential stability of nonautonomous neural network models with continuous distributed delays," *Appl. Math. Comput.*, vol. 219, pp. 9296–9307, 2013.

[5] X. Wu, Y. Tang, and W. Zhang, "Stability analysis of stochastic delayed systems with an application to multi-agent systems," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4143–4149, 2016.

[6] H. Chen, J. Liang, and Z. Wang, "Pinning controllability of autonomous boolean control networks," *Sci. China Inf. Sci.*, vol. 59, p. 070107, 2016.

[7] X. Wu, Y. Tang, and W. Zhang, "Input-to-state stability of impulsive stochastic delayed systems under linear assumptions," *Automatica*, vol. 66, pp. 195–204, 2016.

[8] R. Rakkiyappan, S. Dharani, and Q. Zhu, "Synchronization of reaction-diffusion neural networks with time-varying delays via stochastic sampled-data controller," *Nonlinear Dyn.*, vol. 79, no. 1, pp. 485–500, 2015.

[9] X. Yang, J. Cao, and Z. Yang, "Synchronization of coupled reaction-diffusion neural networks with time-varying delays via pinning-impulsive controller," *SIAM J. Control Optim.*, vol. 51, no. 5, pp. 3486–3510, 2013.

[10] Y. Sheng, H. Zhang, and Z. Zeng, "Synchronization of reaction-diffusion neural networks with dirichlet boundary conditions and infinite delays," *IEEE Trans. Cybern.*, vol. 47, no. 10, pp. 3005–3017, 2017.

[11] A. Arbi, F. Chérif, C. Aouiti, and A. Touati, "Dynamics of new class of hopfield neural networks with time-varying and distributed delays," *Acta Math. Sci.*, vol. 36, no. 3, pp. 891–912, 2016.

[12] E. Arslan, M. S. Ali, and S. Saravanan, "Finite-time stability of stochastic cohen-grossberg neural networks with markovian jumping parameters and distributed time-varying delays," *Neural Process. Lett.*, vol. 46, no. 1, pp. 71–81, 2017.

[13] Q. Zhu and J. Cao, "Exponential stability of stochastic neural networks with both markovian jump parameters and mixed time delays," *IEEE Trans. Syst., Man, Cybern. B*, vol. 41, no. 2, pp. 341–353, 2011.

[14] Y. Sheng and Z. Zeng, "Synchronization of stochastic reaction-diffusion neural networks with dirichlet boundary conditions and unbounded delays," *Neural Netw.*, vol. 93, pp. 89–98, 2017.

[15] T. Wei, L. Wang, and Y. Wang, "Existence, uniqueness and stability of mild solutions to stochastic reaction-diffusion cohen-grossberg neural networks with delays and wiener processes," *Neurocomputing*, vol. 239, pp. 19–27, 2017.

[16] L. Wang, "Global well-posedness and stability of the mild solutions for a class of stochastic partial functional differential equations (in chinese)," *Sci. Sin. Math.*, vol. 47, pp. 371–382, 2017.

[17] L. V. Hien, D. T. Son, and H. Trinh, "On global dissipativity of nonautonomous neural networks with multiple proportional delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 1, pp. 225–231, 2018.

[18] Y. Lv, W. Lv, and J. Sun, "Convergence dynamics of stochastic reaction-diffusion recurrent neural networks with continuously distributed delays," *Nonlinear Anal. RWA*, vol. 9, no. 4, pp. 1590–1606, 2008.

[19] L. Wang, *Delayed Recurrent Neural Networks*. Beijing, China: Science Press, 2008.

[20] L. Gawarecki and V. Mandrekar, *Stochastic Differential Equations in Infinite Dimensions: with Applications to Stochastic Partial Differential Equations*. Springer Berlin Heidelberg, 2011.

[21] L. Wang and D. Xu, "Global exponential stability of hopfield reaction-diffusion neural networks with time-varying delays," *Sci. China Inf. Sci.*, vol. 46, no. 6, pp. 466–474, 2003.

[22] R. Jahanipur, "Stochastic functional evolution equations with monotone nonlinearity: existence and stability of the mild solutions," *J. Differential Equations*, vol. 248, no. 5, pp. 1230–1255, 2010.

[23] L. Arnold, *Stochastic Differential Equations: Theory and Applications*. New York: John Wiley & Sons, 1974.

[24] D. Xu, X. Wang, and Z. Yang, "Existence-uniqueness problems for infinite dimensional stochastic differential equations with delays," *J. Appl. Anal. Comput.*, vol. 2, no. 4, pp. 449–463, 2012.

[25] Q. Zhang, X. Wei, and J. Xu, "Global exponential stability for nonautonomous cellular neural networks with unbounded delays," *Chaos Solitons Fractals*, vol. 39, no. 3, pp. 1144–1151, 2009.

[26] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic bam neural networks with impulse control and mixed time delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 3, pp. 467–479, 2012.